**Allama Iqbal Open University, Islamabad**

Department of Mathematics

Level: BS | Semester: Autumn 2026

Paper: Linear Algebra (Math-3007) | Maximum Marks: 100 | Pass Marks: 50

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Note: Attempt all questions, all questions carry equal marks.

Assignment: 01

## Systems of Linear Equations and Gaussian Elimination

### Question 1:

(a) Solve the following system using Gaussian elimination. Show each row operation and classify the solution set using Theorem 1.1 (Trichotomy of Solutions). [10]

$$\begin{cases} x + y + z = 6 \\ 2x + y + z = 8 \\ x + 2y + z = 9 \end{cases}$$

(b) Consider the system [10]

$$\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 3 \\ x - y = 0 \end{cases}$$

- Form the augmented matrix and row-reduce it.
- Decide whether the system is consistent. If it is not, identify the contradictory row and explain using Theorem 1.1.
- Using Theorem 1.4, explain in one sentence what this means about whether $b$ lies in Col(A).

## Matrix Algebra and Matrix Operations

### Question 2:

(a) Let [10]

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

- Compute AB and BA. Are they equal? What does this illustrate about matrix multiplication?
- Find $A^{-1}$ using the 2 x 2 inverse formula (Theorem 2.3). Verify that $A\,A^{-1} = I$.
- Find $(AB)^{-1}$ using Theorem 2.4 (inverse of a product): $(AB)^{-1} = B^{-1}A^{-1}$.

(b) Let $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

- Compute det(C). Is C invertible? Cite Theorem 5.5 / the Invertible Matrix Theorem (Theorem 2.5).
- Justify your answer using two further IMT conditions (for example: whether the columns are linearly independent, and whether Cx = 0 has only the trivial solution).
- Give one nonzero vector in Nul(C).

## Vector Spaces, Subspaces, and Basis

### Question 3:

(a) Let $v_1 = (1,0,1), \quad v_2 = (0,1,1), \quad v_3 = (1,1,0)$

- Form the matrix A whose columns are these three vectors and compute det(A).
- Decide whether the set is linearly independent, using Theorem 3.5 (independence via row reduction / nonzero determinant).
- Does the set form a basis for R^3? Cite the Basis Theorem (Theorem 3.8).

(b) Let $H = \{(x, y, z) \in R^3 : x - y = 0\}$. [10]

- Show that H is a subspace of R^3 using the Subspace Test (Theorem 3.2).
- Find a basis for H and state dim(H).
- Write a general vector of H as a linear combination of your basis vectors.

## Linear Transformations

### Question 4:

(a) Determine whether each map is a linear transformation. If it is, prove it from Definition 4.1. If it is not, give a specific counterexample (failure of additivity, homogeneity, or T(0) = 0). [10]

$$T(x, y) = (2x, x + y)$$

$$S(x, y) = (x + 1, y)$$

(b) The linear map $T: R^2 \to R^2$ has standard matrix $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

- Is T one-to-one? Justify using Theorem 4.3 (equivalently, compute det(A) or Nul(A)).
- Is T onto? Justify using Theorem 4.4.
- Let S be reflection through y = x, with standard matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the standard matrix of S composed with T (that is, ST), using Theorem 4.5.

## Determinants

### Question 5:

(a) Let, $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

- Compute det(A) using the fact that A is upper triangular (Theorem 5.3).
- Confirm the value by cofactor expansion along the first column (Theorem 5.1).
- Use Theorem 5.5 to decide whether A is invertible, and compute $\det(A^{(-1)})$.

(b) Consider the 2 x 2 system: $\begin{cases} 3x + y = 7 \\ x + 2y = 4 \end{cases}$

- Solve the system by Cramer's Rule (Theorem 5.8). Show both determinants.
- Let $M = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ be the coefficient matrix. Interpret |det(M)| as the area of the parallelogram spanned by the columns (Theorem 5.6).
- Does the linear map $x \mapsto Mx$ preserve or reverse orientation? Cite Theorem 5.7.

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Assignment: 02

## Eigenvalues, Eigenvectors, and Diagonalization

### Question 1:

(a) Let, [10]

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

(i) Find the characteristic polynomial and the eigenvalues (Theorem 6.1). (Hint: A is triangular.)

(ii) Find one eigenvector for each eigenvalue.

(iii) Verify that $tr(A)$ equals the sum of the eigenvalues and $\det(A)$ equals their product (Theorem 6.6). Is A diagonalizable? Why? (Theorem 6.3 / distinct eigenvalues.)

(b) A two-state Markov chain has column-stochastic transition matrix [10]

$$P = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$$

(i) Show that $\lambda = 1$ is an eigenvalue of P and find a corresponding eigenvector.

(ii) Normalize that eigenvector so that its entries are nonnegative and sum to 1 (the stationary distribution).

(iii) Using Theorem 6.4, state the long-run behavior of $x_{(k+1)} = P\,x_k$.

## Orthogonality, Inner Products, and Least Squares

### Question 2:

(a) Apply the Gram-Schmidt process to the ordered set [10]

$$\{(1,1),(2,0)\}$$

to obtain an orthogonal basis of $R^2$. Then normalize to obtain an orthonormal basis. Show the inner products you subtract.

(b)

Fit a line $y = \beta_0 + \beta_1 x$ to the data points (1,1), (2,3), (3,4), (4,6) using least squares. **[10]**

- Set up the matrix equation $A\boldsymbol{\beta} = \boldsymbol{y}$.
- Write the normal equations (Theorem 7.6).
- Solve for $\widehat{\boldsymbol{\beta}}$.
- Compute the residual vector r and $\|r\|^2$. In one sentence, cite the Best Approximation Theorem (Theorem 7.5).

## Singular Value Decomposition, Matrix Factorizations, and PCA

### Question 3:

(a) Compute the full SVD of $A = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}$. [10]

- Form $A^TA$ and find its eigenvalues and orthonormal eigenvectors (these give V and the singular values).
- For each positive singular value, compute $u_i = \left(\frac{1}{\sigma_i}\right) A\,v_i$ and complete U to an orthonormal basis of $R^2$.

- Write $A = U\Sigma V^T$ and check the product. What does a zero singular value tell you about rank(A)?

(b) A centered data matrix $X \in R^{\{6\times3\}}$ has singular values 10, 5, and 2. [10]

- Compute $\|X\|_F^2$ and the proportion of variance explained by the first two principal components (Theorem 8.4).
- Using Eckart-Young (Theorem 8.3), find $\|X - X_2\|_F$ for the best rank-2 approximation.
- How many principal components can carry nonzero variance? If one feature is an exact copy of another, what happens to the smallest singular value?

## Advanced Applications (Machine Learning, Computer Graphics, NLP, Quantum Computing, and Graphs)

### Question 4:

(a) The Hadamard gate is [10]

$$H = \left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

- Compute $H\begin{bmatrix}1\\0\end{bmatrix}$.
- Find the two measurement probabilities. (They should sum to 1.)
- Compute $H^T H$ and conclude that H is unitary (in this real case, orthogonal). Why must quantum gates be unitary?

(b) A two-page web has links only from page 1 to page 2 and from page 2 to page 1. The column-stochastic hyperlink matrix is [10]

$$M = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

- With damping d = 0.85, form the Google matrix $G = 0.85\,M + \left(\frac{0.15}{2}\right)11^T$ and write G with numerical entries.
- Starting from $x_0 = \begin{bmatrix}1\\0\end{bmatrix}$, compute one Power Method step G x_0 (Theorem 6.5).
- Why does damping prevent the ranking from oscillating forever on this two-cycle graph? Relate your answer to the unique dominant eigenvalue $\lambda = 1$ of G (Unit 9).

## Integrative Problem [Connecting Units 6-9]

### Question 5:

(a) A data matrix $X \in R^{\{200\times10\}}$ has rank 8. [10]

- Use Rank-Nullity (Theorem 3.9 / Unit 3) to find $dim(Nul(X))$. What does this say about the 10 features?
- If PCA is applied to X, what is the maximum number of principal components with nonzero variance, and why (Unit 8)?
- The sample covariance $S = \left(\frac{1}{199}\right)X^T X$ is symmetric. Which theorem guarantees real eigenvalues and orthogonal eigenvectors, and why does that matter for PCA? (Theorem 7.7)

(b) An iterative method uses an update matrix $B$ whose eigenvalues are $0.4, -0.3, and\ 0.2$. A ratings matrix $R \in R^{\{500\times40\}}$ is stored via a rank-5 SVD. [10]

- Using Theorem 6.4, does $B^k v$ go to 0 $as\ k$ tends to infinity? Explain using the spectral radius.
- In the sense of Eckart-Young (Theorem 8.3), what is guaranteed about this $rank - 5$ approximation of R?