Allama Iqbal Open University, Islamabad

Department of Mathematics

Level: BS | Semester: Spring 2026

Paper: Linear Algebra (Math-3007) | Maximum Marks: 100; Pass Marks: 50

Note: Attempt all questions; all questions carry equal marks.

Eigenvalues, Eigenvectors, and Diagonalization

Question 1:

(a) For the matrix **[10]**

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

- Find the eigenvalues and eigenvectors.
- Verify that $\mathrm{tr}(A) = \sum \lambda_i$ and $\det(A) = \prod \lambda_i$ (Theorem 6.6).
- Find a matrix P that diagonalizes A, and the corresponding diagonal matrix D.

(b) A 3×3 matrix has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 5$. The eigenspace for $\lambda = 2$ has dimension 1. **[10]**

- Is the matrix diagonalizable? Explain using the Diagonalization Theorem (Theorem 6.3).
- What is the geometric multiplicity of $\lambda = 2$?
- What is the algebraic multiplicity of $\lambda = 2$?
- Is the matrix defective? Explain.

Orthogonality, Inner Products, and Least Squares

Question 2:

(a) Apply the Gram-Schmidt process to the set

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\} = \{(1,1,1), (1,0,1), (0,1,1)\}$$

To find an orthonormal basis for the subspace spanned by these vectors. **[10]**

(b) Fit a line $y = \beta_0 + \beta_1 x$ to the data points (1,1), (2,3), (3,4), (4,6)Usingleast squares. **[10]**

- Set up the matrix equation $A\boldsymbol{\beta} = \boldsymbol{y}$.
- Write the normal equations (Theorem 7.6).
- Solve for $\hat{\boldsymbol{\beta}}$.
- Compute the residual vector and the sum of squared residuals.

Singular Value Decomposition (SVD), Matrix Factorizations, and Principal Component Analysis (PCA)

Question 3:

(a) Compute the full SVD of the matrix **[10]**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Show all steps including:

- Computing $A^T A$ and its eigenvalues/eigenvectors to find V and Σ
- Computing U using the formula $\boldsymbol{u}_i = \frac{1}{\sigma_i} A \boldsymbol{v}_i$
- Completing U to an orthonormal basis if necessary
- Verifying $A = U\Sigma V^T$

(b) A 1000×1000 image matrix A has singular values such that the sum of all squared singular values is 10^6, and the sum of the top 50 squared singular values is 9.8×10^5. **[10]**

- Using the Eckart-Young Theorem (Theorem 8.3), compute the relative Frobenius norm error for a rank-50 approximation.
- Compute the storage savings (in terms of floating-point numbers) if using rank-50 SVD compression compared to storing the full 1000×1000 matrix.
- Explain why natural images can typically be compressed effectively using low-rank approximations.

Advanced Applications (ML, CG, NLP etc.)

Question 4:

(a) Consider a simple neural network with two linear layers (no activation):

$$\boldsymbol{h}_1 = W_1 \boldsymbol{x}, \quad \boldsymbol{h}_2 = W_2 \boldsymbol{h}_1$$

where $W_1 \in \mathbb{R}^{3\times 5}$ and $W_2 \in \mathbb{R}^{2\times 3}$. **[10]**

- Show that this two-layer network is equivalent to a single linear layer. What is the single matrix?
- Explain, citing Worked Example 4.4, why adding a third linear layer without nonlinearity would not increase expressive power.
- Propose a simple nonlinearity that would make the network more expressive and explain why it works.

(b) A graph has the following adjacency matrix (for an undirected graph with 4 nodes): **[10]**

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

- Compute the degree matrix D and the graph Laplacian $L = D - A$.
- Find the eigenvalues and eigenvectors of L.
- What is the multiplicity of the eigenvalue 0, and what does this tell you about the graph's connected components? (Cite the standard graph theory fact from Unit 9.)
- Identify the Fiedler vector (eigenvector for the second smallest eigenvalue) and explain how it could be used for spectral clustering.

Integrative Problem [Connecting Multiple Units]

Question 5:

(a) A data matrix $X \in \mathbb{R}^{500\times 20}$ has rank 15. **[10]**

- Use the Rank-Nullity Theorem (Unit 3) to find $\dim(\text{Nul}(X))$.
- What does this imply about redundancy among the 20 features? (Unit 3)
- If PCA is applied to this data, what is the maximum number of principal components that can carry nonzero variance? Why? (Unit 8)
- The covariance matrix $S = \frac{1}{n-1}X^TX$ is symmetric. What theorem guarantees that it has real eigenvalues and orthogonal eigenvectors? State the theorem and explain its significance for PCA. (Units 7-8)

(b) A recommendation system's ratings matrix $R \in \mathbb{R}^{10000\times 500}$ is approximated as

$$R \approx U_{20}\Sigma_{20}V_{20}^T$$

(rank 20 approximation). **[10]**

- Using the Eckart-Young Theorem (Unit 8), explain in what precise sense this is the "best possible" rank-20 approximation.
- Estimate the storage saved (in floating-point numbers) compared to storing R in full.
- Explain how a new user with very few ratings could still be assigned a reasonable latent vector. Connect this to the dimensionality of the latent space. (Units 3, 8)

- Suppose the system uses gradient descent to fit the low-rank factors. If the update matrix has eigenvalues all less than 1 in magnitude, what does this imply about the convergence of the optimization? (Unit 6, Theorem 6.4)

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Systems of Linear Equations and Gaussian Elimination

Question 1:

(a) Solve the following system using Gaussian elimination. Show every row operation and classify the solution set (unique, no solution, or infinitely many). **[10]**

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 5x_2 + x_3 = 4 \\ 3x_1 + 7x_2 + 3x_3 = 5 \end{cases}$$

(b) For the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & 1 & -1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$$

Determine whether the system is consistent. If it is consistent, express the solution set in parametric vector form. If not, explain why using Theorem 1.4 (consistency and column space). **[10]**

Matrix Algebra and Matrix Operations

Question 2:

(a) For the matrices **[10]**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

compute:

- AB and BA
- Show that $(AB)^T = B^T A^T$
- Find $(AB)^{-1}$ using Theorem 2.4 (inverse of a product)

(b) Use the Invertible Matrix Theorem (IMT) to determine whether the matrix below is invertible. Justify your answer using at least **three** different IMT conditions (state each condition explicitly). **[10]**

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Vector Spaces, Subspaces, and Basis

Question 3:

(a) Let,

$$H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \mid a + 2b - c = 0 \text{ and } 2a - b + c = 0 \right\}$$

Show that H is a subspace of $\mathbb{R}^3$ using the Subspace Test (Theorem 3.2). Find a basis for H and state $\dim(H)$. **[10]**

(b) For the matrix **[10]**

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ -1 & -2 & 1 & -3 \end{bmatrix}$$

find:

- $\text{Nul}(A)$ with a basis
- $\text{Col}(A)$ with a basis
- Verify the Rank-Nullity Theorem (Theorem 3.9)

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Linear Transformations

Question 4:

(a) Determine whether the following maps are linear transformations. For each, either prove linearity using Definition 4.1 or provide a counterexample showing failure of additivity or homogeneity. **[10]**

- $T_1(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 2x_1)$
- $T_2(x_1, x_2) = (x_1^2, x_1 x_2, x_2^2)$
- $T_3(x_1, x_2) = (x_1 + 1, x_2)$

(b) A linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ has standard matrix **[10]**

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

- Is T one-to-one? Justify using Theorem 4.3.
- Is T onto? Justify using Theorem 4.4.
- Find the standard matrix of the composition $T \circ S$ where $S: \mathbb{R}^2 \to \mathbb{R}^3$ has a standard matrix

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Determinants

Question 5:

(a) Compute the determinant of the following matrix using **row reduction** (not cofactor expansion). Show every row operation and track its effect on the determinant. **[10]**

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

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(b) A triangle in $\mathbb{R}^2$ has vertices $P_1 = (1,1)$, $P_2 = (4,2)$, and $P_3 = (2,5)$. **[10]**

- Use the determinant formula from Unit 5 (Real-World Problem 1) to compute the area of this triangle.
- Determine whether the vertices are ordered clockwise or counterclockwise using the sign of the determinant.
- Explain how the determinant sign relates to orientation, citing Theorem 5.7.